

# Nearly Self-Consistent Disk-Bulge-Halo Models for Galaxies

Konrad Kuijken<sup>1</sup> and John Dubinski<sup>2</sup>

<sup>1</sup> *Kapteyn Institute, PO Box 800, 9700 AV Groningen, The Netherlands*

<sup>2</sup> *Lick Observatory, University of California, Santa Cruz, CA 95064, USA*

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## ABSTRACT

We describe methods for setting up self-consistent disk-bulge-halo galaxy models. The bulge and halo distribution functions (DF) are functions of  $E$  and  $L_z$  only. The halo’s flattening and rotation can be specified. The disk DF is a function of  $E$  and  $L_z$  and a third “integral”,  $E_z$ , the vertical energy, which is approximately conserved in a warm disk with vertical extent. The models also have finite extent making them suitable for N-body simulation. A simulation of a sample model shows that in practice the models are very close to equilibrium making them ideal for experiments on instabilities in galactic disks. We also present a sequence of models closely resembling the Milky Way mass distribution with 5 exponential scale radii and varying halo mass and radial extent.

**Key words:** Galaxies: Kinematics and Dynamics – Galaxy: Kinematics and Dynamics

## 1 INTRODUCTION

This paper describes a set of semi-analytic models for the phase-space distribution functions of axisymmetric disk galaxies. They contain three components, corresponding to a disk, bulge and halo. The distribution functions are simple functions of three integrals of motion: the two analytic ones (energy and angular momentum about the axis of symmetry) and one approximate integral which describes the vertical motions in the disk component. These distribution functions yield a unique density for each component in any given potential: since all three components affect each other gravitationally, a numerical solution of the Poisson equation is required for self-consistent models.

Setting up a stable 3 dimensional disk for N-body experiments is a difficult task. Strategies to handle this problem have included adiabatically growing the disk mass distribution in a self-consistent halo/bulge model (e.g. Barnes 1988), treating the the halo and bulge as a static background (e.g. Sellwood & Merritt 1994; Quinn, Hernquist & Fullagar 1993), or directly solving the Jeans equations (under a suitable Ansatz) for the complete system of the disk, bulge and halo to find the velocity dispersions (Hernquist 1993), and realizing these dispersions with (typically) Gaussian distributions. In a recent study, Sellwood & Merritt (1994) used the DF calculated by Kalnajs (1976) of the Kuzmin-Toomre disk to set up the radial and azimuthal velocities, but this method is restricted to that class of models, and they did use the Jeans equations for the vertical velocities.

While these methods are useful for disks with small  $N$  where the discreteness noise is dominant, as  $N$  increases the approximate initial conditions introduce subtle transient behaviour which can interfere with the interpretation of experiments. Adiabatic growth of the disk, or relaxation from non-equilibrium initial conditions have the further disadvantage that the relaxed initial conditions are not under perfect control of the experimenter. It is for example not uncommon to observe outward propagating rings of overdensity from the warmer disk center (where the Gaussian approximation is the poorest), and transients in the disk velocity dispersion which can change the initial values by  $\sim 20\%$  before the particles settle into equilibrium (Hernquist, Mihos, and Walker, private communication). As a result, though these models relax quickly to equilibrium, the velocity dispersion profiles change from the input values. The density profiles usually do not change significantly. While such subtle effects do not matter in experiments in which a disk will be strongly perturbed (say by a merger), they can be significant in investigations of the rather subtle instabilities which arise from the inherent fragility of a cool disk. Examples of these problems are spiral structure formation (Sellwood & Carlberg 1984), bar instabilities (e.g. Hernquist and

Weinberg 1992), bending instabilities in counter-streaming disks (e.g. Sellwood & Merritt 1994; Merritt & Sellwood 1994), disk warping (Dubinski & Kuijken 1995), and disk heating by satellite accretion or other tidal perturbations (e.g. Quinn, Hernquist & Fullager 1993). Simulations with  $N$  in the millions are now becoming possible with greater computing power. The subtle transients at start-up due to various approximations which were previously drowned out by discreteness, become correspondingly more bothersome. A cleaner set of initial conditions are desirable.

In this paper we describe the construction of our models. In §2, we present the disk distribution function (DF) of the disk, bulge and halo and describe a method for calculating the potential of the self-consistent model. The bulge and halo have the DF of King (1966) models and their flattened generalization, the lowered Evans distributions (Kuijken & Dubinski 1994). These DFs depend only on the classical integrals of motion, and can be combined straightforwardly. The disk distribution function is a generalization to three dimensions of the planar model devised by Shu (1969) (see also Kuijken & Tremaine 1992), similar to the construction by Binney (1987), and allows almost arbitrary specification of the radial variation of density and velocity dispersion. Since two-integral distribution functions for disks are not realistic (Oort 1965), we employ an approximate third integral. §3 contains N-body demonstrations and tests of equilibrium for a specific model. In §4, we present a sequence of models with different halo extent suitable for describing the Milky Way. Finally, we discuss the advantages and limitations of these models in §5 and summarize our results.

## 2 DISTRIBUTION FUNCTION

Our strategy is an extension of the technique we employed in the construction of the lowered Evans models (Kuijken & Dubinski 1994), and earlier applied by e.g., Prendergast & Tomer (1970) and Rowley (1988). The starting point is a chosen analytic form for the DF, written in terms of known integrals of motion. Any such DF represents an equilibrium model in any gravitational potential which respects these integrals, with space density explicitly determined by this potential. We can therefore construct self-gravitating models by requiring that the potential and the density be related by Poisson's equation. This last step must generally be done numerically.

In order of increasing complexity, we now detail the DF's of the three galactic components.

### 2.1 The Bulge Distribution Function

For the bulge DF we take a King model (King 1966). This DF has the form

$$f_{\text{bulge}}(E) = \begin{cases} \rho_b (2\pi\sigma_b^2)^{-3/2} \exp[(\Psi_o - \Psi_c)/\sigma_b^2] \{\exp[-(E - \Psi_c)/\sigma_b^2] - 1\} & \text{if } E < \Psi_c, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

It depends on the three parameters  $\Psi_c$  (the cutoff potential of the bulge),  $\rho_b$  (approximately the central bulge density, ignoring the effects of the DF truncation) and  $\sigma_b$ , which governs the velocity dispersion of the bulge component.  $\Psi_0$  is the gravitational potential at the center of the model.

In what follows, we will normally choose  $\sigma_b < \sigma_0$  and  $\Psi_c < 0$  to make the bulge more centrally condensed, and more radially confined, than the halo (the latter has a cutoff at zero energy).

### 2.2 The Halo Distribution Function

We use the DF of a lowered Evans model (Kuijken & Dubinski 1994) for the halo. This DF is a truncation at finite energy of the models discovered by Evans (1993) for the flattened logarithmic potential. It takes the form

$$f_{\text{halo}}(E, L_z^2) = \begin{cases} [(AL_z^2 + B) \exp(-E/\sigma_0^2) + C] [\exp(-E/\sigma_0^2) - 1] & \text{if } E < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The density corresponding to this DF is given by eq. 9 of Kuijken & Dubinski (1994), and is repeated here for completeness:

$$\begin{aligned} \rho_{\text{halo}}(R, \Psi) &= \frac{1}{2} \pi^{3/2} \sigma_0^3 (AR^2 \sigma_0^2 + 2B) \operatorname{erf}(\sqrt{-2\Psi}/\sigma_0) \exp(-2\Psi/\sigma_0^2) \\ &\quad + (2\pi)^{3/2} \sigma_0^3 (C - B - AR^2 \sigma_0^2) \operatorname{erf}(\sqrt{-\Psi}/\sigma_0) \exp(-\Psi/\sigma_0^2) \\ &\quad + \pi \sqrt{-2\Psi} [\sigma_0^2 (3A\sigma_0^2 R^2 + 2B - 4C) + \frac{4}{3} \Psi (2C - A\sigma_0^2 R^2)], \end{aligned} \quad (3)$$

where  $\operatorname{erf}(x) = 2\pi^{-1/2} \int_0^x \exp(-t^2) dt$  is the usual error function. The halo DF has five free parameters: the potential well depth  $\Psi_0$ , the velocity and density scales  $\sigma_0$  and  $\rho_1$ , the halo core radius  $R_c$  and the flattening parameter  $q$ , (the last three of these contained within the parameters  $A$ ,  $B$ , and  $C$ ). For convenience, we have defined a characteristic halo radius  $R_a$  which replaces the density scale  $\rho_1$  of Kuijken & Dubinski (1994):

$$R_a = \left( \frac{3}{2\pi G \rho_1} \right)^{1/2} \sigma_0 e^{\Psi_0/2\sigma_0^2}; \quad (4)$$

it is roughly the radius at which the halo rotation curve, if continued at its  $R = 0$  slope, would reach the value  $2^{1/2}\sigma_0$ .

Arbitrary amounts of rotation can be added to the halo model by splitting the DF into parts with positive and negative  $L_z$ .

### 2.3 The Disk Distribution Function

In the construction of a realistic three-integral disk distribution function, the issue of a third integral cannot be evaded (as it was for the bulge and halo components). For it is an observed fact that in the solar neighbourhood (e.g., Wielen 1974) and in the disks of other galaxies (Bottema 1993) the vertical and radial dispersions are different, which is not possible in any DF that depends only on energy and angular momentum. The simplest approximate third integral in an axisymmetric disk system is the energy in the vertical oscillations,  $E_z \equiv \Psi(R, z) - \Psi(R, 0) + \frac{1}{2}v_z^2$ . It is quite well conserved along nearly circular orbits which have no large radial or vertical excursions. We will use this quantity as third integral for the disk DF in our models. More sophisticated possible integrals are discussed by Kent and de Zeeuw (1991).

Armed with a third integral, the disk distribution function can be constructed by extending the planar DF discussed by Shu (1969) and Kuijken & Tremaine (1992) into the vertical dimension, similar to the DF constructed by Binney (1987). Thus, we have

$$f_{\text{disk}}(E_p, L_z, E_z) = \frac{\Omega(R_c)}{(2\pi^3)^{1/2}\kappa(R_c)} \frac{\tilde{\rho}_d(R_c)}{\tilde{\sigma}_R^2(R_c)\tilde{\sigma}_z(R_c)} \exp \left[ -\frac{E_p - E_c(R_c)}{\tilde{\sigma}_R^2(R_c)} - \frac{E_z}{\tilde{\sigma}_z^2(R_c)} \right]. \quad (5)$$

Here,  $E_p \equiv E - E_z$  is the energy in planar motions,  $L_z$  is the specific angular momentum about the axis of symmetry,  $R_c$  and  $E_c$  are the radius and energy of a circular orbit with angular momentum  $L_z$ , and  $\Omega$  and  $\kappa$  are the circular and epicyclic frequencies at radius  $R_c$ . The density corresponding to this DF is obtained by integrating over the three velocity components. The  $v_R$  and  $v_z$  integrals are straightforward, leaving the  $v_\phi$ -integral:

$$\rho_{\text{disk}}(R, z) = \int_0^\infty \left\{ \left[ dv_\phi \equiv dR_c \left( \frac{R_c \kappa(R_c)^2}{2R\Omega(R_c)} \right) \right] \frac{2\tilde{\rho}_d(R_c)\Omega(R_c)}{(2\pi)^{1/2}\tilde{\sigma}_R(R_c)\kappa(R_c)} \times \exp \left[ -\frac{\Psi(R, 0) - \Psi(R_c, 0)}{\tilde{\sigma}_R^2(R_c)} - \left( \frac{R_c^2}{R^2} - 1 \right) \frac{v_c^2(R_c)}{2\tilde{\sigma}_R^2(R_c)} - \frac{\Psi(R, z) - \Psi(R, 0)}{\tilde{\sigma}_z^2(R_c)} \right] \right\}. \quad (6)$$

In the  $z = 0$  plane, this expression reduces to  $\tilde{\rho}_d(R)$  with fractional error  $O(\tilde{\sigma}_R^2/v_c^2)$ , and to the same order the radial velocity distribution is Gaussian with dispersion  $\tilde{\sigma}_R(R)$  (see Kuijken & Tremaine 1992). The essence of the construction in eq. 5 is the replacement of the radius  $R$  (which is not an integral of motion) by the epicyclic radius  $R_c$  (which is a function of angular momentum, and therefore is conserved along orbits). In warm disks, in which excursions from circular orbits are small but not negligible, this parametrization still provides a good starting point for constructing a DF with given radial density and velocity dispersion profiles. The vertical structure of this disk is approximately isothermal, with the scale height set by the vertical velocity dispersion  $\tilde{\sigma}_z(R_c)$  and the vertical potential gradient.

Observed large disk galaxies have vertical light profiles similar to the  $\text{sech}^2(z)$ -dependence expected for a vertically isothermal, self-gravitating sheet (van der Kruit & Searle 1981): therefore, as long as the disk is mainly confined in  $z$  by its own gravity, the DF of equation 5 will be a reasonable model.

In any gravitational potential, we can adjust the ‘tilde’ functions  $\tilde{\rho}$ ,  $\tilde{\sigma}_R$  and  $\tilde{\sigma}_z$  to the desired disk characteristics. In this paper, we arrange for the disk density to be approximately radially exponential and truncated:

$$\rho_{\text{disk}}(R, z) = \frac{M_d}{2\pi R_d^2 z_d} e^{-R/R_d} \text{erfc} \left( \frac{r - R_{\text{out}}}{2^{1/2}\delta R_{\text{out}}} \right) \exp \left[ \ln \text{sech}^2(1) \frac{\Psi_z(R, z)}{\Psi_z(R, z_d)} \right] \quad (7)$$

Here  $M_d$  is a parameter which is close to the mass of the disk unless the disk is severely truncated or the vertical structure is far from  $\text{sech}^2(z/z_d)$ . The vertical density of these disks is constructed to depend exponentially on the vertical potential  $\Psi_z(R, z) \equiv \Psi(R, z) - \Psi(R, 0)$ , and to drop from the mid-plane value by a factor  $\text{sech}^2(1)$  at a height of  $z_d$ , similar to the behaviour of a constant thickness isothermal sheet.

Given a total potential for the model, we then set the disk tilde functions in the disk DF as follows. In the limit of very small velocity dispersions these functions are the actual mid-plane density and velocity dispersions. We first choose the function  $\tilde{\sigma}_R(R_c)$ , approximately determining the radial velocity dispersion in the disk.  $\tilde{\rho}$  and  $\tilde{\sigma}_z$  are then iteratively adjusted so that the density on the mid-plane and at height  $z = z_d$  agree with those of equation 7. It turns out that, at least for the models described in the remainder of this paper, this recipe yields a DF which has a space density close to that given by equation 7.

### 2.4 Calculation of the Combined Potential

The distribution functions for the various galaxy components all imply a unique volume density in a given potential. To

**Figure 1.** Representative contours of the resulting radial and vertical forces for a  $l = 32$  pure multipole expansion (*dotted line*) and  $l = 8$  expansion of the residue plus  $\Psi_{\text{disk}}$  (*solid line*). Higher order expansions of the residue plus  $\Psi_{\text{disk}}$  give the same result as the  $l = 8$  expansion. The forces derived from the  $l = 32$  pure multipole expansion still have significant errors at small  $z$  and intermediate  $R$ .

construct a self-gravitating model, we need to find the potential in which the combined density is also the one implied by Poisson's equation, i.e.

$$\nabla^2 \Psi(R, z) = 4\pi G[\rho_{\text{disk}}(R, \Psi, \Psi_z) + \rho_{\text{bulge}}(\Psi) + \rho_{\text{halo}}(R, \Psi)]. \quad (8)$$

We solve equation 8 using a spherical harmonic expansion, following Prendergast & Toomer (1970), with two significant modifications. First, we have found that the disk density obtained by integrating the disk DF (eq. 5) over all velocities is close to the value given by eq. 7 (this was, after all, what the disk DF was designed to do). We therefore use this latter expression in the solution of Poisson's equation: this change avoids the single integral that would have to be calculated numerically each time the disk density was needed. Second, and more importantly, for realistically thin disks a spherical harmonic expansion is not very efficient, since high-order terms must be taken before a good approximation can be obtained. Moreover, zero-thickness disks, which form a very regular limit physically, require a prohibitive number of terms in the series. We therefore construct an analytic potential which represents the high-frequency terms correctly, and only fit harmonics to the residue.

A possible analytic 'high harmonics' disk potential  $\Psi_{\text{disk}}^\dagger$  is obtained by vertically integrating the disk density twice (i.e. by solving Poisson's equation for the disk component ignoring the radial gradient terms). Assuming that the vertical profile of the disk is that of a self-gravitating, plane-parallel sheet, this approach yields

$$\Psi_{\text{disk}}^\dagger = 4\pi G \rho_d(R) z_d^2 \ln \cosh(z/z_d) \quad (9)$$

There are two reasons why this potential, as written, is not suitable for removing the high harmonics. The first is that the density corresponding to this potential,  $\nabla^2 \Psi_{\text{disk}}^\dagger / 4\pi G$ , does not converge to 0 at large distances from the origin. The residual density can therefore not be fitted conveniently with spherical harmonic coefficients. We resolve this problem by replacing cylindrical radius  $R$  by spherical radius  $r$  in equation 9: since the radial density profile has a cutoff, the resulting potential and its corresponding density will indeed converge at large radii, while at small  $z$  the potential still models well the higher frequency terms of the disk. The second problem is that, even after rewriting the equation in terms of  $r$ , the potential has discontinuous derivatives at zero radius, if the disk density is radially exponential. We therefore smoothly round off the potential in the central disk scale length, by patching on a quartic function in the central region. At small radii, the spherical harmonic expansion is in any case better able to follow the disk density, since the projected angle subtended by the disk,  $z_d/R$ , only gets unmanageably small at large radii.

The high-frequency disk potential we use is then

$$\Psi_{\text{disk}}(r, z) = \frac{M_d z_d}{4\pi R_d^2} \ln \cosh(z/z_d) \times \begin{cases} \frac{1}{4} e^{-1} [7 - 4(r/R_d)^2 + (r/R_d)^4] & \text{for } r < R_d, \\ \frac{1}{2} \operatorname{erfc}\left(\frac{r - R_{\text{out}}}{2^{1/2} \delta R_{\text{out}}}\right) e^{-r/R_d} & \text{otherwise.} \end{cases} \quad (10)$$

The corresponding volume density is rather involved, but can be obtained using the result

$$\nabla^2 f(r) \ln \cosh z = f''(r) \ln \cosh z + 2 \frac{f'(r)}{r} (z \tanh z + \ln \cosh z) + f(r) \operatorname{sech}^2 z, \quad (11)$$

where once again it should be remembered that  $r$  is the spherical radius. Note that the final term reproduces the disk density of a  $\operatorname{sech}^2(z)$  disk to  $O(z/R)^2$ , and that the other terms have traded a vertical derivative for a radial one, and so are subdominant in thin disks. Note that it is important to have the resolution to follow accurately the *radial* variation in this density near the truncation radius, where the  $f''$  term now magnifies the change in disk density.

Experience shows that, for a disk scale height of  $0.15R_d$  (i.e., exponential scale height  $R_d/13.3$  far from the disk plane) the residual density can be well-fitted with a series truncated at  $l = 8$ , whereas even a series expansion to  $l = 32$  for the disk density alone does not give satisfactory results (Figure 1).

### 3 A SAMPLE MODEL

The advantage of galaxy models built from distribution functions is that the kinematics are fully specified. In this section we describe how this kinematic information can be used to set up equilibrium initial conditions for N-body simulations of disk-bulge-halo galaxies, and provide some sample results.

**Figure 2.** Rotation curve of a sample model.**Table 1.** Galaxy Model Parameters.

Model	DISK					BULGE			HALO				
	$M_d$	$R_d$	$R_t$	$z_d$	$\delta R$	$\Psi_c$	$\sigma_b$	$\rho_b$	$\Psi_o$	$\sigma_0$	$q$	$C$	$R_a$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
Sample	1.00	1.0	5.0	0.15	0.3	-2.0	0.50	10.0	-4.0	1.00	0.9	0.1	0.5
MW-A	0.87	1.0	5.0	0.10	0.5	-2.3	0.71	14.5	-4.6	1.00	1.0	0.1	0.8
MW-B	0.87	1.0	5.0	0.10	0.5	-2.9	0.71	14.5	-5.2	0.96	1.0	0.1	0.8
MW-C	0.87	1.0	5.0	0.10	0.5	-3.7	0.71	14.5	-6.0	0.93	1.0	0.1	0.8
MW-D	0.87	1.0	5.0	0.10	0.5	-4.7	0.71	14.5	-7.0	0.92	1.0	0.1	0.8

(1) disk mass, (2) disk scale radius, (3) disk truncation radius, (4) disk scale height, (5) disk truncation width, (6) bulge cut off potential, (7) bulge velocity dispersion, (8) bulge central density, (9) halo central potential (10) halo velocity dispersion, (11) halo potential flattening, (12) halo concentration,  $C = R_c^2/R_K^2$ , (13) characteristic halo radius.

### 3.1 Setting up an N-body Realization

We can generate an N-body realization of a galaxy by randomly sampling from the DF's for each component. The bulge and the halo are straightforward to generate since the systems are nearly spherical and the velocity ellipsoids are nearly isotropic. The methods are described in detail in Kuijken & Dubinski (1994). A particle's position is first determined by sampling from the density distribution. With this position, one can find the local maximum of the DF (at  $(v_x, v_y, v_z) = (0, 0, 0)$ ) and then use the acceptance-rejection technique to find a velocity. This involves selecting the three components of the velocity at random from a velocity sphere with radius equal to the escape velocity. A random value,  $f_{ran}$ , of the DF is also chosen between 0 and the local maximum. If  $f_{ran}$  is less than the value of the DF at the chosen velocity then the velocity is accepted, otherwise it is rejected and another attempt is made.

Sampling from the disk DF is slightly more troublesome, since the disk is thin and the local velocity maximum of the disk DF must be found at each point (it is not at  $\mathbf{v} = 0$  as for the bulge and halo). As before, we sample first from the density distribution to find particle positions and then from the DF to find velocities. Since the disks are generally warm, the maximum of the velocity distribution occurs at a point where the azimuthal component of velocity  $v_\phi$  is less than the local circular velocity,  $v_{circ}$ .  $v_R$  and  $v_z$  are zero at the local maximum. We find the local velocity maximum for each particle position using standard methods (Press et al. 1993) and then use the acceptance-rejection technique as before to select a velocity.

### 3.2 An N-body Realization

We set up an N-body realization of a sample model to test the validity of the equilibrium. This galaxy model has disk:bulge mass ratio of 4:1 and a halo:disk mass ratio of 1.5:1 within 5 disk scale lengths. The total mass of the halo is about 10 times the disk mass with the halo extending to 40 disk scale lengths (see Table 1 and 2 for the model parameters and resulting properties). The rotation curve is fairly flat out to a radius of 10 disk scale lengths (Figure 2).

The disk is warm with a Toomre  $Q = 1.7$  at the disk half mass radius. The value of  $Q$  is fairly constant throughout the disk, though rising both in the center within 1 scale length and near the edge where the surface density tapers off to zero (Figure 3).

We performed various simulations to assess the validity of our approximate disk DF and the stability of the overall system. We used a tree code for all of the simulations (Barnes & Hut 1986; Hernquist 1987; Dubinski 1988). The critical opening

**Table 2.** Galaxy Model Properties.

Model	DISK			BULGE		HALO	
	$M$	$\sigma_{r,0}$	$R_e/R_d$	$M$	$R_e/R_d$	$M$	$R_e/R_d$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Sample	0.94	0.50	5.6	0.29	1.7	9.6	44.9
MW-A	0.82	0.47	6.0	0.42	1.0	5.2	21.8
MW-B	0.82	0.47	6.0	0.43	1.0	9.6	30.1
MW-C	0.82	0.47	6.0	0.43	1.0	19.8	44.0
MW-D	0.82	0.47	6.0	0.43	1.0	37.0	72.8

(1) disk mass, (2) disk central radial velocity dispersion,  $\sigma_r, 0$ , (3) disk radial extent (radius where density drops to zero), (4) bulge mass, (5) bulge radial extent, (6) halo mass, (7) halo radial extent.

**Figure 3.** Toomre’s stability parameter  $Q = \sigma_r \kappa / 3.36 G \Sigma$  versus the radius**Figure 4.** The disk surface density profile as a function of time for the 4 simulations as labelled.

angle was set to  $\theta = 0.9$  with forces between cells and particles calculated to quadrupole order. The particle softening radius was set to 0.025 disk scale lengths (or 0.17 disk scale heights). The orbital time of the model at the disk half mass radius is 13 units. We set the leapfrog timestep to 0.1 units and ran the models to  $t = 96$  or approximately 7.5 orbital times. We set up 4 simulations:

- (i) a disk of 40000 test particles orbiting in the derived system potential;
- (ii) a gravitating disk of 40000 particles with a static bulge and halo potential;
- (iii) a disk, bulge, and halo with 40000, 10000, and 50000 particles respectively; and
- (iv) a disk, bulge and halo with 80000, 20000, and 200000 particles.

For all of the models, we followed the time evolution of the disk surface density profile,  $\Sigma(R)$ , the disk velocity dispersion profile averaged in rings, and the disk scale height versus radius. For the bulge and halo, we calculated the spherically averaged density profiles.

The first simulation provides a simple check of our DF, since the test particles should maintain a constant density and velocity dispersion profile in the total potential of the model. This test is most critical for the disk since we have approximated a third integral with the vertical energy,  $E_z$  and it is not guaranteed that this quantity is conserved sufficiently well to maintain a decent equilibrium. For our sample model, we find that the approximation works very well. There is essentially no change in the density profile (Figure 4) and velocity dispersion profiles (Figure 5).

In the second simulation, we made the disk ‘live’ while maintaining a static bulge and halo potential. We thereby avoid external perturbations from a noisy bulge and halo potential (which we will see below can significantly heat the disk). The surface density profile is nearly constant, once again deviating only slightly from the initial conditions. The disk is initially cool enough to suffer from spiral arm instabilities and this manifests itself as a gradual rise in  $\sigma_R$  and  $\sigma_\phi$  throughout the disk (Figure 5) (e.g. Sellwood & Carlberg 1984, Toomre & Kalnajs 1991). This self-heating is unavoidable in cool disks, even when set up from a DF which is formally in equilibrium. The vertical velocity dispersion,  $\sigma_z$ , remains fairly constant except at the edge of the disk where it is poorly sampled by particles and discreteness noise is large, and near the center where the disk is hot and the assumed constancy of  $E_z$  along orbits built in to the DF breaks down. Nevertheless,  $\sigma_z$  does not show any significant initial transients over most of the disk suggesting that the disk DF represents a starting point very close to equilibrium. We do not expect spiral arm formation to heat the disk vertically, since the modes grow in the disk plane and couple weakly to vertical oscillations (e.g., Jenkins & Binney 1990). We also plotted the r.m.s. height of disk particles averaged in rings (Figure 6). There is a slight increase with time in the average scale height which may arise from either the particle softening, which would weaken the local disk self-gravity and allow it to puff up, or from gradual heating by the disk’s own discreteness.

The final two simulations, in which the halo and bulge are also live, show the strong effect of a coarse-grained halo on the disk evolution. A particle halo adds a significant source of disk heating (Hernquist 1993) when the number of halo particles is too small. The halo is effectively composed of massive black holes that bombard the disk as in the model of Lacey & Ostriker (1985). The continuous bombardment of halo particles heats the disk at a rate  $\sigma^2 = (\sigma_0^2 + Dt)$ , where  $D$  is proportional to the black hole mass and local halo density. The disk velocity ellipsoid therefore changes and the vertical scale height increases. This effect is strongest at the disk center where the halo density is the greatest, and is weakened as the number of halo particles is increased, in agreement with the Lacey-Ostriker results. The graininess of the halo can also excite bending instabilities especially in the outer disk where the disk density is low. The disks in simulation 3 do indeed exhibit bending instabilities that resemble a flag waving in the breeze. This behaviour was less apparent in the larger simulation 4. Again as in simulation 2 gravitational softening can also cause the disk to puff up.

Figure 5 shows the velocity dispersion profiles for the simulations with particle bulges and halos. We can see that there is significant heating for the smaller  $N$  model over and above the heating of the disk in the static halo potential. The disk heating for the large  $N$  halo is significantly smaller with the  $\sigma_z$  increasing only at disk radii with  $R > 3R_d$ . The disk scale height also does not grow as much for the larger  $N$  model in accord with expected heating rates (Figure 6). The models are apparently converging to a smooth equilibrium limit as we increase  $N$ .

Unlike the disk, the bulge and halo maintain their integrity during the simulations, as expected since they are derived

**Figure 5.** The velocity dispersion profiles as a function of time for the 4 simulations as labelled. To avoid overlap,  $\sigma_\phi$  and  $\sigma_z$  are offset by 0.5 and 1.0 dex

**Figure 6.** The disk scale height  $\langle z^2 \rangle^{1/2}$  as a function of time for the 4 simulations as labelled.

**Figure 7.** The bulge and halo density profiles for largest simulation as a function of time. The bulge relaxes to a slightly lower density for  $r < 0.1$  probably as a result of the gravitational softening, but overall the density profiles remain constant.

from DF's with exact integrals of motion in the given potential. Figure 7 shows the constancy of the density profiles of the models at 3 times. The slight drop in the density in the bulge within  $r < 0.1$  can be attributed to the particle softening radius  $\epsilon = 0.025$ . The reduced gravity from the softening allows the bulge to expand a little at the center and therefore drop in density. Despite this minor defect, the bulge and halo stay in equilibrium.

At late times (5-8 orbital times), a modest bar begins to develop within 1 scale radius in the completely live models. The bar forms sooner in the smaller  $N$  models. The seeds of these bar instabilities probably originate in the discreteness noise of the halo also observed by Hernquist (private communication). The onset of bar formation can only be delayed by increasing  $N$ .

In summary, a disk DF with  $E_z$  as an approximate third integral results in a disk-bulge-halo galaxy model very close to formal equilibrium. We can therefore follow subtle changes in the disk structure and kinematics resulting from a variety of instabilities.

In the next section, we find a set of model parameters which produce a mass model closely resembling the Galaxy.

#### 4 MODELS OF THE MILKY WAY

In Table 1, we present the parameters for generating a sequence of 4 models, MW-A,B,C, and D, which have mass distributions and rotation curves closely resembling those of the Milky Way within 5 scale radii. The disk and bulge mass distributions are the same for each model with mass and extent of the halo increasing through the sequence (Table 2). The halos are all chosen with  $q = 1.0$ , though they are slightly squashed in the self-consistent galaxy models. Model MW-D has the halo with the largest mass and has the most realistic representation of the outer galaxy. These models were found by trial and error and renormalized so that the flat portion of the rotation curve had  $V_c \approx 1.0$ . The contributions to the radial acceleration in the solar neighbourhood ( $R = 1.8R_d$ ) from disk, bulge and halo are comparable in these models, as found by Kuijken & Gilmore (1989) in their study of the local disk surface density. The natural units for length, velocity, and mass for these dimensionless models are  $R_d = 4.5$  kpc,  $V = 220$  km s<sup>-1</sup>, and  $M = 5.1 \times 10^{10} M_\odot$ . The central velocity dispersion was chosen so that the observed radial velocity dispersion of 42 km s<sup>-1</sup> at the solar radius ( $R = 1.8R_d$ ) would be reproduced in the model. Figures 8 and 9 show the rotation curves out to 5 scale radii and 50 scale radii respectively.

We tested the stability of the models against bar formation and found that they were not immediately unstable. A small bar within 1 scale radius did form after four orbital times, though increasing the number of particles delayed the formation to later times. It seems that the observed instability was due to the discreteness of the simulation rather than the intrinsic mass distribution. In the limit of large  $N$ , these particular models are probably stable against bar formation. The fact that we observe a bar in the galactic center (e.g. Weinberg 1992) might imply a larger disk:halo mass radius within the solar circle than these models assume.

#### 5 SUMMARY

We have described a method for setting up self-consistent galaxy models with a disk, bulge and halo where each component is described by a separate phase-space distribution function (DF). The halo DF depends on particle energy  $E$  and angular momentum  $L_z$ , and has the form of a lowered Evans model (Kuijken & Dubinski 1994), resulting in a flattened, finite radius model. The bulge DF is a King model (King 1966) which is a function only of  $E$ . We introduce a new DF for the disk, a version of the planar DF of Shu (1969) modified to include vertical structure. It is a function of  $E$ ,  $L_z$  and  $E_z$ , the vertical energy. The disk DF is approximate in that the third “integral”,  $E_z$ , introduced to describe the vertical motion, is not conserved exactly. Nevertheless, in practice the disk DF yields a model very close to equilibrium. The three components respond to the total gravitational potential of the model. The main advantages of these models over previous ones are:

**Figure 8.** The rotation curves for the Milky Way models, MW-A, B, C, and D, showing contributions from the disk, bulge and halo in the inner regions within  $R < 5R_d$ .

**Figure 9.** The rotation curves for the Milky Way models, MW-A, B, C, and D, showing contributions from the disk, bulge and halo in the outer regions out to  $R = 50R_d$ .

- (i) The flattening and rotation of the halo can be specified in the Lowered Evans model DF, which allows examination of new problems where these parameters are important such as the dynamics of galactic warps and satellite orbital evolution.
- (ii) The disk does not suffer from additional transient adjustments at start up beyond the expected (and unavoidable) spiral and bar instabilities. The potential of the system and velocity ellipsoids (particularly those of the disk) will therefore not change significantly from the initial state. These models are therefore useful for studying problems involving the effects of small perturbations on the disk such as disk heating by sinking satellites or external tidal fields.
- (iii) Since the DF is known explicitly, many quantities (line profiles, for example) can be calculated directly without the need for simulation.

These same techniques can be generalized to models with different bulge and halo DF's and disks with different density distributions. Of course, equilibrium does not imply stability, and it is by no means guaranteed that these models are free from spiral and bar instabilities. In practice, though, models can be found which are apparently stable over many rotation periods. Disk heating by the halo is troublesome, and ultimately can only be avoided by going to very large  $N$  ( $\sim 10^6$ ) if we wish to evolve a system for a Hubble time.

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